OCU-PHUS 179 (Revised)

October 2001

Fermion propagator in out of equilibrium quantum-field system and the Boltzmann equation

A. Niégawa*

Department of Physics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, JAPAN

Abstract

We aim to construct from first principles a perturbative framework for studying nonequilibrium quantum-field systems that include massless Dirac fermions. The system of our concern is quasiuniform system near equilibrium or nonequilibrium quasistationary system. We employ the closed-time-path formalism and use the so-called gradient approximation. Essentially no further approximation is introduced. We construct a fermion propagator, with which a well-defined perturbative framework is formulated. In the course of construction of the framework, we obtain the generalized Boltzmann equation (GBE) that describes the evolution of the number-density functions of (anti)fermionic quasiparticles.

11.10.Wx, 12.38.Mh, 12.38.Bx

I. INTRODUCTION

Ultrarelativistic heavy-ion-collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC) have begun and will soon start at the CERN Large Hadron Collider (LHC)

*Electronic address: niegawa@sci.osaka-cu.ac.jp

in an anticipation of producing a quark-gluon plasma (QGP) [see, e.g., [1,2]]. The QGP to be produced is an expanding nonequilibrium system. Studies of the QGP as such have just begun.

In previous papers, a perturbative framework has been formulated *from first principles* for dealing with out-of-equilibrium complex-scalar field system [3], $O(N)$ linear-sigma system [4], and the system that includes gauge bosons [5]. In this paper, we take up the out-of-equilibrium quantum-field theories that include Dirac fermions. Keeping in mind that the masses of light quarks may be ignored in QGP, we treat massless fermions. [Generalization to massive ones is straightforward.] Through similar procedure as in [3–5], we construct the fermion propagator and, thereby, frame a perturbation theory. Essentially, only approximation we employ is the so-called gradient approximation (see below). We use the closed-time-path (CTP) formalism [6–8] of nonequilibrium statistical quantum-field theory.

Throughout this paper, we are interested in quasiuniform systems near equilibrium or nonequilibrium quasistationary systems. Such systems are characterized (cf., e.g., [9]) by two different spacetime scales: microscopic or quantum-field-theoretical and macroscopic or statistical. The first scale, the microscopic-correlation scale, characterizes the range of radiative correction to reactions taking place in the system while the second scale measures the relaxation of the system. For a weak-coupling theory, in which we are interested in this paper, the former scale is much smaller than the latter scale. A well-known intuitive picture [9] for dealing with such systems is to separate spacetime into many "cells" whose characteristic size, L^μ ($\mu = 0, ..., 3$), is in between the microscopic and macroscopic scales. It is assumed that the correlation between different cells is small, so that microscopic or elementary reactions can be regarded, to a good approximation, as taking place in a single cell. On the other hand, in a single cell, prominent relaxation phenomena do not take place. This intuitive picture may be implemented as follows. Let $S(x, y)$ be a propagator. For a system of our concern, $S(x, y)$, with $x^\mu - y^\mu$ [$|x^\mu - y^\mu| \lesssim L^\mu$] fixed, does not change appreciably within a single cell. Thus, the mid-point $X^\mu \equiv (x^\mu + y^\mu)/2$ may be used as a label for the spacetime cells and is called the macroscopic spacetime coordinates or a slow

variable [10]. On the other hand, relative spacetime coordinates $x^\mu - y^\mu$, which is called a fast variable, are responsible for describing microscopic reactions taking place in a single spacetime cell. We introduce a Wigner transformation (Fourier transformation with respect to the relative coordinates $x - y$ with $(x + y)/2$ held fixed):

$$S(x, y) = \int \frac{d^4P}{(2\pi)^4} \, e^{-iP\cdot(x-y)} \, S(X; P) \tag{1.1}$$

(with $P^\mu = (p^0, \mathbf{p})$). The Wigner function $S(X; P)$ depends on X only weakly. The self-energy part $\Sigma(x, y)$ enjoys a similar property. We shall freely use $S(x, y)$ or $S(X; P)$ [$\Sigma(x, y)$ or $\Sigma(X; P)$], which we simply write S [Σ] whenever obvious from the context.

In the following, there often appears a "product" of functions:

$$[F \cdot G](x, y) = \int d^4z \, F(x, z)G(z, y), \tag{1.2}$$

For the Wigner transform of the function $[F \cdot G](x, y)$, we use the gradient approximation,

$$\begin{aligned} [F \cdot G](X; P) &= \int d^4(x - y) \, e^{iP\cdot(x-y)}[F \cdot G](x, y) \\ &\simeq F(X; P)G(X; P) \\ &\quad -\frac{i}{2} \{F(X; P), \; G(X; P)\}_{\text{P.B.}} \, , \end{aligned} \tag{1.3}$$

where $X = (x + y)/2$ and

$$\begin{aligned} \{F, \; G\}_{\text{P.B.}} \equiv & \frac{\partial F(X; P)}{\partial X^\mu} \frac{\partial G(X; P)}{\partial P_\mu} \\ & -\frac{\partial F(X; P)}{\partial P_\mu} \frac{\partial G(X; P)}{\partial X^\mu} . \end{aligned} \tag{1.4}$$

We refer to the first term on the right-hand side (RHS) of Eq. (1.3) as the *leading part (term)* and to the second term, which includes derivative ∂_X, as the *gradient* or *nonleading part (term)*. Crudely speaking, the leading term is the quantity in itself in the spacetime cell labeled by X, while the gradient term represents the effect arising from the (weak) correlation between the spacetime cell labeled by $(x + z)/2$ (see Eq. (1.2)) and the one labeled by $(z + y)/2$.

The perturbative framework to be constructed accompanies the generalized Boltzmann equation (GBE) for the number density of quasiparticles. The framework allows us to compute any reaction rate by using the *reaction-rate formula* deduced in [11]. Substituting the computed net production rates of quasiparticles into the GBE, one can determine the number densities as functions of spacetime coordinates X, which describes the evolution of the system.

For the sake of concrete presentation, we take up QCD. However, the procedure has little to do with QCD and then the framework to be constructed may be used for any theory that includes massless Dirac fermion(s) with almost no modification. The plan of the paper is as follows: In Sec. II, making use of the free quark (fermion) fields in vacuum theory and the quark-distribution function at the initial time ($X^0 = -\infty$), we construct the bare quark propagator, with which the "bare-N scheme" may be constructed. The perturbative calculation on the basis of this scheme yields [3] divergence due to pinch singularities. Then, in Sec. III, we set up the basis for formulating the "physical-N scheme" by introducing a new quark-distribution function and a new "free" quark fields. In Sec. IV, we make up the self-energy-part resummed quark propagator. In Sec. V, imposing the condition on the quark-distribution function that there do not appear large contributions, which stems from the above-mentioned pinch singularities, we construct a "healthy" perturbative framework. It is shown that, on the energy-shell, the condition turns out to be the generalized Boltzmann equation. In Sec. VI, we frame a concrete perturbative framework. Section VII is devoted to summary and discussion, in which comparison with related works is made. Concrete derivation of various formula used in the text is made in Appendices.

II. CLOSED-TIME-PATH FORMALISM

A. Preliminary

We follow the procedure as in [5]. The CTP formalism is reduced to a two-component formalism [7]. Every field, say ϕ, is doubled, $\phi \to (\phi_1, \phi_2)$, and the classical action turns out

$$\int_{-\infty}^{+\infty} dx_0 \int d\mathbf{x} \, [\mathcal{L}(\phi_1(x), ...) - \mathcal{L}(\phi_2(x), ...)] \tag{2.1}$$

with $\mathcal{L}$ the Lagrangian density of the theory under consideration. For definiteness, we take up massless QCD. The gluon sector has already been studied in [5]. In this paper, we deal with the quark sector,

$$\mathcal{L} = i\bar{\psi}^a(x)\partial\!\!\!/ \, \psi^a(x) + ..., \tag{2.2}$$

where "a" is a color index. Perturbation theory is formulated in terms of propagators, vertices, and initial correlations[†]. The aim of the present paper is to construct the quark propagator, which takes the 2×2 matrix form [7],

$$\hat{S}^{ab}_{\alpha\beta}(x, y) \equiv \begin{pmatrix} -i\langle T(\psi^a_1(x))_\alpha (\bar{\psi}^b_1(y))_\beta \rangle & +i\langle (\bar{\psi}^b_2(y))_\beta (\psi^a_1(x))_\alpha \rangle \\ -i\langle (\psi^a_2(x))_\alpha (\bar{\psi}^b_1(y))_\beta \rangle & -i\langle \bar{T}(\psi^a_2(x))_\alpha (\bar{\psi}^b_2(y))_\beta \rangle \end{pmatrix}. \tag{2.3}$$

Here $\langle ... \rangle \equiv \mathrm{Tr}\,(... \, \rho)$ with ρ the density matrix at the initial time, the 'caret' denotes the 2×2 matrix as indicated on the RHS, the suffix α (β) denotes the component of ψ^a_j ($\bar{\psi}^a_j$) $[j = 1, 2]$, and T ($\bar{T}$) is the time-ordering (antitime-ordering) symbol. Noting that the state realized at a heavy-ion collision is color singlet, we restrict ourselves to the case where ρ is color singlet, so that $\hat{S}^{ab}_{\alpha\beta} = \delta^{ab}\hat{S}_{\alpha\beta}$. We also assume that ρ commutes with baryonic charge. [Generalization to the color-nonsinglet case is straightforward.] In the sequel, we drop the color index. At the end of calculation we set $\psi_1 = \psi_2$ and $\bar{\psi}_1 = \bar{\psi}_2$ [7]. Instructive examples of ρ, in the case of scalar field, is given in [9].

[†]Given a density matrix, which characterize the system, the initial correlations are determined [7,10]. In perturbation theory, the initial correlations are treated as vertexes.

B. Quark propagator

We start with computing Eq. (2.3) with $\psi_1 = \psi_2$ ($\equiv \psi$) and $\bar{\psi}_1 = \bar{\psi}_2$ ($\equiv \bar{\psi}$). For ψ and $\bar{\psi}$, we use the plane-wave decomposition with helicity basis in vacuum theory, which, in standard notation, reads

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p}} \sum_{\xi=\pm} \Big[b_\xi(p) u_\xi(p) e^{-iP\cdot x} + d_\xi^\dagger(p) v_\xi(p) e^{iP\cdot x} \Big] \qquad (p^0 = p), \tag{2.4}$$

where ξ ($= \pm$) denotes helicity, and $b_\xi(p)$ [$b_\xi^\dagger(p)$] is the annihilation [creation] operator of a quark [an antiquark] with helicity ξ. Substituting Eq. (2.4) and its adjoint into Eq. (2.3), we compute the Wigner transform, $\hat{S}(X;P)$, of $\hat{S}(x,y)$ [cf. Eq. (1.1)] to the gradient approximation. There emerge four types of terms, which contain $\langle b_\xi^\dagger b_\zeta \rangle$, $\langle d_\xi^\dagger d_\zeta \rangle$, $\langle b_\xi d_\zeta \rangle$, and $\langle b_\xi^\dagger d_\zeta^\dagger \rangle$, respectively. For an equilibrium system, only $\langle b_\pm^\dagger b_\pm \rangle$ and $\langle d_\pm^\dagger d_\pm \rangle$ terms survive, and $\langle b_+^\dagger b_+ \rangle = \langle b_-^\dagger b_- \rangle$ and $\langle d_+^\dagger d_+ \rangle = \langle d_-^\dagger d_- \rangle$ hold. With this observation in mind, we see that, for the systems of our interest, the leading term of $\hat{S}(X;P)$, which we write $\hat{S}_0(X;P)$, contains the "even" combinations $\sum_{\xi=\pm} \langle b_\xi^\dagger b_\xi \rangle$ and $\sum_{\xi=\pm} \langle d_\xi^\dagger d_\xi \rangle$. It turns out that the gradient term $\hat{S}_1(X;P)$ contains also $\sum_{\xi=\pm} \langle b_\xi^\dagger b_\xi \rangle$ and $\sum_{\xi=\pm} \langle d_\xi^\dagger d_\xi \rangle$ together with spacetime (X) derivative. We assume that the term $\hat{S}_2(X;P)$ that contains the "odd" combinations $\sum_{\xi=\pm} \xi \langle b_\xi^\dagger b_\xi \rangle$ and $\sum_{\xi=\pm} \xi \langle d_\xi^\dagger d_\xi \rangle$ and the term $\hat{S}_3(X;P)$ that contains $\langle b_\pm^\dagger b_\mp \rangle$ and $\langle d_\pm^\dagger d_\mp \rangle$ are nonleading and are, at most, of the same order of magnitude as the gradient term $\hat{S}_1(X;P)$. We also assume that the terms with $\langle b_\xi d_\zeta \rangle$ and with $\langle b_\xi^\dagger d_\zeta^\dagger \rangle$ may be ignored. It is a straightforward task to obtain the form of $\hat{S}(X;P)$ for the case where some or all of the above assumptions do not hold.

Thus, $\hat{S}(X;P)$ consists of four terms, the leading term $\hat{S}_0$, the gradient term $\hat{S}_1$, and the nonleading terms $\hat{S}_2 + \hat{S}_3$,

$$\hat{S}(X;P) = \hat{S}_0(X;P) + \hat{S}_1(X;P) + \hat{S}_2(X;P) + \hat{S}_3(X;P). \tag{2.5}$$

The form for $\hat{S}_2$ and $\hat{S}_3$ are displayed in Appendix A. The terms $\hat{S}_0$ and $\hat{S}_1$ reads

$$\hat{S}_0(X;P) = \not{P} \begin{pmatrix} \Delta_R(P) & 0 \\ \Delta_R(P) - \Delta_A(P) & -\Delta_A(P) \end{pmatrix} - f(X;P)\not{P}\left[\Delta_R(P) - \Delta_A(P)\right]\hat{A}_+, \quad (2.6)$$

$$\hat{S}_1(X;P) = -\frac{\epsilon^{0\mu\nu\rho} + \epsilon^{3\mu\nu\rho}}{2(p^0+p^3)} P_\mu \gamma_5 \gamma_\rho \frac{\partial f(X;P)}{\partial X^\nu}\left[\Delta_R(P) - \Delta_A(P)\right]\hat{A}_+. \quad (2.7)$$

Here $\epsilon^{0123} = 1$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and

$$\hat{A}_\pm = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \quad (2.8)$$

$$\Delta_{R(A)}(P) = \frac{1}{P^2 \pm ip_0 0^+},$$

$$f(X;P) = \theta(-p_0) + \epsilon(p_0) N(X;P), \quad (2.9)$$

$$\begin{aligned} N(X;P) &\equiv \frac{1}{2}\Big[\theta(p^0)\left\{N_{++}(X;|p^0|,\tilde{\mathbf{p}}) + N_{--}(X;|p^0|,\tilde{\mathbf{p}})\right\} \\ &\quad +\theta(-p^0)\left\{\bar{N}_{++}(X;|p^0|,-\tilde{\mathbf{p}}) + \bar{N}_{--}(X;|p^0|,-\tilde{\mathbf{p}})\right\}\Big] \\ &\equiv \theta(p^0) n(X;|p^0|,\tilde{\mathbf{p}}) + \theta(-p^0)\bar{n}(X;|p^0|,-\tilde{\mathbf{p}}), \end{aligned} \quad (2.10)$$

with

$$N_{\xi\zeta}(X;|p_0|,\tilde{\mathbf{p}}) \equiv \int d^3q\, e^{-iQ\cdot X}\mathrm{Tr}\left[b_\xi^\dagger(\mathbf{p}-\mathbf{q}/2)\, b_\zeta(\mathbf{p}+\mathbf{q}/2)\,\rho\right] \quad (\xi,\zeta = +,-). \quad (2.11)$$

Here $|p_0| = p$, $\tilde{\mathbf{p}} \equiv \mathbf{p}/p$, and $Q = (q^0, \mathbf{q})$ with $q_0 = \mathbf{q}\cdot\mathbf{p}/p$. $\bar{N}$'s are antiquark counterpart of N's. For the system of our concern, Tr[...] is different from from zero [7,10] only for q small compared with $\mathbf{p}$. It can readily be seen from Eq. (2.11) that when ρ is translationally invariant $N_{\xi\zeta}(X;|p^0|,\tilde{\mathbf{p}})$ is independent of X. In passing, for a charge-conjugation-invariant system, $N(X;P) = N(X;-P)$ holds.

From Eq. (2.11) follows

$$\begin{aligned} &\theta(p^0) P\cdot\partial_X N_{\xi\zeta}(X;|p^0|,\tilde{\mathbf{p}}) \\ &\quad = \theta(-p^0) P\cdot\partial_X \bar{N}_{\xi\zeta}(X;|p^0|,-\tilde{\mathbf{p}}) = 0 \qquad (|p_0| = p). \end{aligned} \quad (2.12)$$

Using this in Eqs. (2.9) and (2.10), we have

$$P \cdot \partial_X f(X;P) = 0 \qquad (|p_0| = p). \tag{2.13}$$

As is obvious from the construction, or as can be directly shown, to the gradient approximation,

$$i\hat{\tau}_3 \partial\!\!\!/_x \hat{S}(x,y) = -i\hat{S}(x,y) \overleftarrow{\partial\!\!\!/}_y \, \hat{\tau}_3 = \hat{1}\, \delta^4(x-y),$$

where $\hat{1}$ is the 2×2 unit matrix and $\hat{\tau}_3$ is the third Pauli matrix. Two equations in Eq. (2.12) are "free Boltzmann equations." One can construct a perturbation theory in a similar manner as in [3]. We call the perturbation theory thus constructed the bare-N scheme, since N obeys the "free Boltzmann equation." Perturbative computation within this scheme yields divergences due to pinch singularities. In [3], how to deal with these divergences is discussed and shown is that the bare-N scheme is equivalent to the physical-N scheme, to which we now turn.

III. CONSTRUCTION OF THE PHYSICAL-N SCHEME

Following the procedure as in [5], we construct a scheme in terms of the number density that is as close as possible to the physical number density. To this end, first of all, we abandon the "free Boltzmann equation" (2.13). This means that f here and in the sequel is different from f $(\equiv f_B)$ in the last section. Then $\hat{S}(X;P;f) \neq \hat{S}(X;P;f_B)$, which means that ψ, and $\bar{\psi}$ in the present (physical-N) scheme differs from the free field in vacuum theory, Eq. (2.4), as employed in the bare-N scheme. [See also the comment below in conjunction with Eq. (3.4).] Specification of f and then also of ψ and $\bar{\psi}$ are postponed until Sec. VI.

Now, $\hat{S}(x,y)$ is not an inverse of $i\hat{\tau}_3 \partial\!\!\!/$ [cf. Eq. (2.1) with Eq. (2.2)]. Straightforward computation using the inverse Wigner transform of $\hat{S}(X;P)$, Eq. (2.5), yields, to the gradient approximation,

$$\begin{aligned} i\hat{\tau}_3 \partial\!\!\!/_x \hat{S}(x,y) &= \hat{1}\,\delta^4(x-y) - \frac{i}{2}\hat{\tau}_3 \hat{A}_+ \int \frac{d^4P}{(2\pi)^4} e^{-iP\cdot(x-y)} \\ &\quad \times \frac{\gamma^0+\gamma^3}{p^0+p^3} P\!\!\!/ (\Delta_R - \Delta_A) P \cdot \partial_X f(X;P). \end{aligned} \tag{3.1}$$

Our procedure of constructing a consistent scheme is as follows: We further modify ψ and $\bar{\psi}$ by adding a suitable $\hat{S}_{\rm add}$ to $\hat{S}$ on the left-hand side (LHS) of Eq. (3.1). The conditions for $\hat{S}_{\rm add}$ to be satisfied are

- $\hat{S}_{\rm add}$ vanishes in the bare-N scheme in Sec. II.

- To the gradient approximation, Eq. (3.1) turns out to

$$\left(i\hat{\tau}_3 \partial\!\!\!/ 1 - \hat{L}_c\right) \cdot \left(\hat{S} + \hat{S}_{\rm add}\right) = \hat{1}\, 1. \tag{3.2}$$

  Here we have used the short-hand notation (1.2). In Eq. (3.2), '1' is the matrix in spacetime whose (x, y)-component is $\delta^4(x-y)$, and $\hat{L}_c$ is some $(4 \times 4) \otimes (2 \times 2)$ matrix function.

It is straightforward to obtain the form of the required $\hat{S}_{\rm add}$:

$$\begin{aligned} \hat{S}_{\rm add}(X; .P) = i\hat{A}_+ & \left[\left(\Delta_R^2 + \Delta_A^2\right) P\!\!\!/ - \frac{\gamma^0 + \gamma^3}{p^0 + p^3} \frac{\mathbf{P}}{P^2} \right] \\ & \times P \cdot \partial_X f(X; P), \end{aligned} \tag{3.3}$$

($\mathbf{P}/P^2$ the principal part of $1/(P^2 \pm i0^+)$) from which we obtain for $\hat{L}_c$ in Eq. (3.2),

$$\begin{aligned} \hat{L}_c(x, y) = L_c \hat{A}_- = i\hat{A}_- & \int \frac{d^4P}{(2\pi)^4} e^{-iP\cdot(x-y)} \frac{\gamma^0 + \gamma^3}{p^0 + p^3} \\ & \times P \cdot \partial_X f(X; P), \end{aligned}$$

with $\hat{A}_-$ as in Eq. (2.8). In obtaining Eq. (3.2) with $\hat{S}_{\rm add}$ as in Eq. (3.3), we have used $\hat{L}_c \cdot \hat{S}_0 \simeq \hat{L}_c \cdot (\hat{S} + \hat{S}_{\rm add})$, since the difference can be ignored to the gradient approximation.

In a similar manner, we find that $(\hat{S} + \hat{S}_{\rm add}) \cdot (-i\hat{\tau}_3 1 \overleftarrow{\partial\!\!\!/} - \hat{L}_c) = \hat{1}\, 1$. Thus we have found that the $(2 \times 2) \otimes (4 \times 4)$ matrix propagator $(\hat{S} + \hat{S}_{\rm add})$ is an inverse of $(i\hat{\tau}_3 \partial\!\!\!/ 1 - \hat{L}_c)$, so that the free action is

$$\begin{aligned} & \int d^4x\, d^4y\, \tilde{\bar{\psi}}(x) \left[i\hat{\tau}_3 \partial\!\!\!/_x \delta^4(x-y) - L_c(x,y)\hat{A}_- \right] \tilde{\psi}(y), \\ & \tilde{\bar{\psi}} = \left(\bar{\psi}_1,\ \bar{\psi}_2\right), \qquad \tilde{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned} \tag{3.4}$$

This is different from the free action in the bare-N scheme, $\int d^4x\, \tilde{\bar{\psi}} i\hat{\tau}_3 \partial\!\!\!/ \tilde{\psi}$, and then the free fields in Eq. (3.4) is different from the free fields employed in the bare-N scheme. Since the term with $L_c(x, y)$ in Eq. (3.4) is absent in the original action, we should introduce the counter action to compensate it,

$$\mathcal{A}_c = \int d^4x\, d^4y\, \tilde{\bar{\psi}}(x) L_c(x, y) \hat{A}_- \tilde{\psi}(y), \tag{3.5}$$

which yields a vertex

$$\begin{aligned} iL_c(x, y)\hat{A}_- &= -\hat{A}_- \int \frac{d^4P}{(2\pi)^4} e^{-iP\cdot(x-y)} \frac{\gamma^0 + \gamma^3}{p^0 + p^3} \\ &\quad \times P \cdot \partial_X f(X; P) \\ &\equiv -i\hat{\Sigma}^{(c)}(x, y). \end{aligned} \tag{3.6}$$

As in [5], it is straightforward task to construct a $\hat{L}_c$-resummed propagator, which we do not reproduce here.

In closing this section, we emphasize that $f(X; P)$ in the present scheme is an arbitrary function, provided that $f(X^0_{in}, \mathbf{X}; P)$ $[= \theta(-p^0) + \epsilon(p^0) N(X^0_{in}, \mathbf{X}; P)]$, with X^0_{in} $(= -\infty)$ the initial time, is a given initial data. We have introduced the counteraction $\mathcal{A}_c$, Eq. (3.5), so as to remain on the original theory. Thus, it cannot be overemphasized that the schemes with different f's are mutually equivalent. If we choose f_B for f, which subjects to the "free Boltzmann equation" (2.13), the scheme reduces to the bare-N scheme in the last section. In Sec. VI, we shall choose f, with which a well-defined perturbation theory is formulated. As will be shown below, it turns out that f is determined order by order in perturbation theory. As a natural assumption, we require that, in the limit $g \to 0$ (g the QCD coupling constant), $f \to f_B$.

IV. RESUMMATION OF THE SELF-ENERGY PART

A. Preliminary

As has been mentioned at the end of last section, when interactions are switched off, the self-energy part $\hat{\Sigma}$ vanishes and Eq. (2.13) holds, $P \cdot \partial_X f(X;P) = 0$. Thus, we suspect that the self-energy part causes the nontrivial evolution of f, $P \cdot \partial_X f(X;P) \neq 0$. Thus, $\hat{\Sigma}$ ties to the gradient part, $\hat{S}_1$, of $\hat{S}$. More precisely, $\hat{\Sigma}$ is of the same order of magnitude as $(\hat{S}_0)^{-1}\hat{S}_1(\hat{S}_0)^{-1}$ and $(\hat{S}_0)^{-1}\hat{S}_{\rm add}(\hat{S}_0)^{-1}$. Hence, in computing $\hat{\Sigma}$ in the approximation under consideration, it is sufficient to keep the leading part (i.e., the part with no X_μ-derivative).

Since we are dealing with the massless quark, the leading part $\hat{\Sigma}(X;P)$ may be decomposed as $\hat{\Sigma} = [\tilde{\sigma}^+ \tilde{\not{P}}_- - \tilde{\sigma}^- \tilde{\not{P}}_+]/2 + \sum_{a=1}^{2} \hat{\sigma}_a \vec{\gamma} \cdot \vec{e}_a(p)$, where

$$\tilde{P}_\pm \equiv (1, \pm\tilde{\mathbf{p}}) \qquad (\tilde{\mathbf{p}} \equiv \mathbf{p}/p), \tag{4.1}$$

and $\vec{e}_1(p)$ and $\vec{e}_2(p)$ are unit vectors being orthogonal to $\mathbf{p}$; $\mathbf{p} \cdot \vec{e}_a(p) = \vec{e}_1(p) \cdot \vec{e}_2(p) = 0$ $(a = 1,2)$. For an equilibrium system, $\hat{\sigma_1} = \hat{\sigma_2} = 0$. Then it is natural to assume that, for the systems of our concern, $\hat{\sigma}_a$ $(a = 1,2)$ can be ignored when compared to the leading part,

$$\hat{\Sigma}(X;P) = \frac{1}{2}\left[\hat{\sigma}^+(X;P)\tilde{\not{P}}_- - \hat{\sigma}^-(X;P)\tilde{\not{P}}_+\right].$$

Generalization to the case where this assumption does not hold is straightforward. It should be noted that $\hat{\Sigma}$ consists of two pieces,

$$\hat{\Sigma} = \hat{\Sigma}^{\rm loop} + \hat{\Sigma}^{(c)},$$

where $\hat{\Sigma}^{\rm loop}$ is the contribution from loop diagrams and $\hat{\Sigma}^{(c)}$ is as in Eq. (3.6). It should be remarked that some $\hat{\Sigma}^{\rm loop}$ contains internal vertex(es) $i\hat{\Sigma}^{(c)}$. [See Sec. VI below.]

Within the gradient approximation, it is sufficient to perform a $\hat{\Sigma}$-resummation for the leading part $\hat{S}_0$. This is because the corrections to other parts due to the resummation are of higher order. Thus, for $\hat{S}_1 - \hat{S}_3$ and $\hat{S}_{\rm add}$, one can use the formulae in the bare-N scheme in Sec. II [see the argument at the end of Sec. III]. In particular, for f in $\hat{S}_1 - \hat{S}_3$ and $\hat{S}_{\rm add}$, one can use f_B. Now we introduce, as usual, the "standard form" (see Eq. (1.2)) [12,5]

$$\left[\hat{B}_L \cdot \hat{S}_{\rm diag} \cdot \hat{B}_R\right](x,y), \tag{4.2}$$

$$\hat{S}_{\rm diag} = {\rm diag}(i\partial\!\!\!/ \Delta_R,\, -i\partial\!\!\!/ \Delta_A),$$

$$\hat{B}_L = \begin{pmatrix} 1 & -f \\ 1 & 1-f \end{pmatrix}, \qquad \hat{B}_R = \begin{pmatrix} 1-f & -f \\ 1 & 1 \end{pmatrix}, \tag{4.3}$$

where f $(= f(x,y))$ is the inverse Wigner transform of $f(X;P)$ and $1(x,y) = \delta^4(x-y)$. Computing Eq. (4.2) to the gradient approximation, we obtain

$$\begin{aligned}\hat{\mathcal{S}}(x,y) &\equiv \hat{S}_0(x,y) + \hat{S}_{\rm add}(x,y) \\ &= \left[\hat{B}_L \cdot \hat{S}_{\rm diag} \cdot \hat{B}_R\right](x,y) + i\int \frac{d^4P}{(2\pi)^4} e^{-iP\cdot(x-y)} \\ &\quad \times \left[\partial\!\!\!/_X f(X;P) - \frac{\gamma^0+\gamma^3}{p^0+p^3} P\cdot\partial_X f(X;P)\right] \\ &\quad \times \frac{\mathbf{P}}{P^2}\hat{A}_+. \end{aligned} \tag{4.4}$$

It is obvious that one can freely include gradient part(s) into the "resummed part." For convenience, we include the gradient part $\hat{S}_{\rm add}$, Eq. (3.3), and take $\hat{\mathcal{S}}$ in Eq. (4.4) as the "resummed part."

It is to be noted that, from Eqs. (2.6) and (3.3) follows[‡]

$$\sum_{i,\,j=1}^{2} (-)^{i+j} \mathcal{S}_{ij} = \sum_{i,\,j=1}^{2} (-)^{i+j} \left[S_{ij} + (S_{\rm add})_{ij}\right] = 0. \tag{4.5}$$

B. Self-energy-part resummed propagator

A $\hat{\Sigma}$-resummed propagator $\hat{G}$ obeys the Schwinger-Dyson (SD) equation:

$$\hat{G} = \hat{\mathcal{S}} + \hat{\mathcal{S}}\cdot\hat{\Sigma}\cdot\hat{G} = \hat{\mathcal{S}} + \hat{G}\cdot\hat{\Sigma}\cdot\hat{\mathcal{S}}. \tag{4.6}$$

We recall that $\hat{G}$ obeys [7] the same relation as Eq. (4.5), $\sum_{i,\,j=1}^{2}(-)^{i+j}G_{ij} = 0$. Using this and Eq. (4.5) in Eq. (4.6), we obtain

---

[‡]Here and in the following, (ij) element of a (2×2) matrix $\hat{M}$ is denoted by M_{ij}.

$$\sum_{i,j=1}^{2} \Sigma_{ij} = 0. \tag{4.7}$$

Procedure of solving Eq. (4.6) is given in Appendix B. The result for $\hat{G}(X;P)$ takes the form

$$\begin{aligned}
&\hat{G}(X;P) \\
&\quad \simeq \begin{pmatrix} G_R(X;P) & 0 \\ G_R(X;P) - G_A(X;P) & -G_A(X;P) \end{pmatrix} \\
&\qquad - \hat{A}_+ \left[G_R(X;P) - G_A(X;P) \right] f(X;P) \\
&\qquad + \hat{A}_+ G_K(X;P).
\end{aligned} \tag{4.8}$$

$G_{R(A)}$ consists of the leading piece and the nonleading piece,

$$G_{R(A)}(X;P) \simeq G^{(0)}_{R(A)}(X;P) + G^{(1)}_{R(A)}(X;P), \tag{4.9}$$

The leading piece reads

$$G^{(0)}_{R(A)}(X;P) = \frac{1}{2} \left[g^{+}_{R(A)}(X;P) \tilde{\not{P}}_+ + g^{-}_{R(A)}(X;P) \tilde{\not{P}}_- \right], \tag{4.10}$$

$$g^{\pm}_{R(A)} = \frac{1}{p^0 \mp p \mp \sigma^{\pm}_{R(A)}}, \tag{4.11}$$

where $\tilde{P}_\pm$ is as in Eq. (4.1) and $\sigma^\tau_{R(A)} = \sigma^\tau_{11} + \sigma^\tau_{12(21)}$. Although the nonleading piece $G^{(1)}_{R(A)}$ can be ignored to the gradient approximation, we have displayed it in Appendix B.

G_K in Eq. (4.8) reads

$$G_K \simeq G^{(0)}_K + G^{(1)}_K, \tag{4.12}$$

$$\begin{aligned}
G^{(0)}_K = -\frac{i}{2} \sum_{\tau=\pm} \Big[\Big(& \tilde{P}_\tau \cdot \partial_X f - i\tau \left(\sigma^\tau_K\right)^{\mathrm{loop}} \\
& + \tau \left\{ \mathrm{Re}\sigma^\tau_R, \ f \right\}_{\mathrm{P.B.}} \Big) g^\tau_R g^\tau_A \tilde{\not{P}}_\tau \Big],
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
G^{(1)}_K = \frac{i}{4} \sum_{\mathcal{H}=R,\,A} \sum_{\tau=\pm} \Big[\Big(& \tilde{P}_\tau \cdot \partial_X f \\
& + \tau \left\{ \sigma^\tau_{\mathcal{H}}, \ f \right\}_{\mathrm{P.B.}} \Big) \left(g^\tau_{\mathcal{H}}\right)^2 \Big] \tilde{\not{P}}_\tau
\end{aligned}$$

$$-\frac{i}{16}\frac{1}{p}\sum_{\tau=\pm}\Big[\tau\tilde{\not{P}}_\tau\left(\gamma_\perp^i\partial_i f\right)\tilde{\not{P}}_{-\tau}\Big(g_R^+ - g_A^+ + g_R^- \\ - g_A^- - 2i\,g_R^\tau g_A^{-\tau}\mathrm{Im}(\sigma_R^+ - \sigma_R^-)\Big)\Big]\,, \tag{4.14}$$

where $\gamma_\perp^i \equiv \gamma^i - (\vec{\gamma}\cdot\tilde{\mathbf{p}})\tilde{p}^i$ and

$$\left(\sigma_K^\pm\right)^{\mathrm{loop}} \equiv [1-f]\left(\sigma_{12}^\pm\right)^{\mathrm{loop}} + f\left(\sigma_{21}^\pm\right)^{\mathrm{loop}}. \tag{4.15}$$

In obtaining Eq. (4.14), use has been made of $\left(\sigma_A^\pm\right)^* = \sigma_R^\pm$, which can straightforwardly be proved [7].

V. GENERALIZED BOLTZMANN EQUATION

A. Energy-shell and physical number densities

For later use, referring to Eq. (4.10) with Eq. (4.11), we define the energy-shell for "normal modes" through

$$\mathrm{Re}\left[g_R^\pm(X;P)\right]^{-1}_{p^0=\pm\omega_\pm^n(X;\pm\mathbf{p})} \\ = \left[p^0 \mp p \mp \mathrm{Re}\sigma_R^\pm(X;P)\right]_{p^0=\pm\omega_\pm^n(X;\pm\mathbf{p})} = 0. \tag{5.1}$$

It is well known [2] that, in equilibrium quark-gluon plasma, "abnormal modes" called plasmino appears for soft p $(= O(gT))$ [g the QCD coupling constant and T the temperature]. The energy-shell of such modes, if any, is defined through

$$\mathrm{Re}\left[g_R^\pm(X;P)\right]^{-1}_{p^0=\mp\omega_\pm^a(X;\mp\mathbf{p})} \\ = \left[p^0 \mp p \mp \mathrm{Re}\sigma_R^\pm(X;P)\right]_{p^0=\mp\omega_\pm^a(X;\mp\mathbf{p})} = 0. \tag{5.2}$$

Useful formulae that hold on the energy-shell are displayed in Appendix C.

In order to obtain the expression for physical number densities, we start with computing charge and momentum densities,

$$\begin{aligned}\mathrm{Tr}\left[j^0(x)\rho\right] &= \mathrm{Tr}\left[\bar{\psi}(x)\gamma^0\psi(x)\,\rho\right] \\ &= -\frac{i}{2}\mathrm{Tr}\left[\gamma^0\left(G_{21}(x,x)+G_{12}(x,x)\right)\rho\right],\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathrm{Tr}\left[\vec{\mathcal{P}}(x)\rho\right] &= -i\mathrm{Tr}\left[\psi^\dagger(x)\nabla\psi(x)\,\rho\right] \\ &= -\frac{1}{2}\mathrm{Tr}\left[\gamma^0\nabla\left(G_{21}(x,x)+G_{12}(x,x)\right)\,\rho\right].\end{aligned} \tag{5.4}$$

We first compute the leading contribution to $\mathrm{Tr}\,[j^0(x)\rho]$. Substituting the leading parts of G_{21} and of G_{12} (cf. Eq. (4.8)) and using Eq. (4.10), we obtain

$$\begin{aligned}\mathrm{Tr}\left[j^0(x)\rho\right] &= i\int\frac{d^4P}{(2\pi)^4}\left[\left(g_+^R(x;P)+g_-^R(x;P)\right)-\mathrm{c.c.}\right] \\ &\qquad\qquad\times[2f(x;P)-1].\end{aligned}$$

From this, we see that $\mathrm{Tr}\,[j^0(x)\rho]$ weakly depends on x. The narrow-width approximation, $Im\sigma_\pm^R \to \mp i0^+$ yields

$$\begin{aligned}\mathrm{Tr}\left[j^0(x)\rho\right] \simeq 2\int\frac{d^3p}{(2\pi)^3}\Big[&Z_+^n(x;\omega_+^n(\mathbf{p}),\mathbf{p})n(x;\omega_+^n(\mathbf{p}),\mathbf{p})-Z_-^n(x;\omega_-^n(\mathbf{p}),\mathbf{p})\bar{n}(x;\omega_-^n(\mathbf{p}),\mathbf{p}) \\ &+Z_-^a(x;\omega_-^a(\mathbf{p}),\mathbf{p})n(x;\omega_-^a(\mathbf{p}),\mathbf{p})-Z_+^a(x;\omega_+^a(\mathbf{p}),\mathbf{p})\bar{n}(x;\omega_+^a(\mathbf{p}),\mathbf{p})\Big]+\ldots.\end{aligned} \tag{5.5}$$

Here n and $\bar{n}$ are as in Eq. (2.10), and '...' stands for the contribution from $2f-1 \ni -\epsilon(p^0)$ [cf. Eq. (2.9)], which is the vacuum-theory contribution corrected by the medium effect. Z's in Eqs. (5.5) are the wave-function renormalization factors, Eqs. (C.1) and (C.4) in Appendix C. The first (last) two contributions on the RHS come from the "normal modes" ("abnormal modes") of quasiparticles. If there are several normal and/or abnormal modes, summation should be taken over all modes. The factor '2' in Eqs. (5.5) comes from the spin degrees of freedom.

In a similar manner, we obtain for the momentum density,

$$\begin{aligned}\mathrm{Tr}\left[\vec{\mathcal{P}}(x)\rho\right] \simeq 2\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\Big[&Z_+^n(x;\omega_+^n(\mathbf{p}),\mathbf{p})n(x;\omega_+^n(\mathbf{p}),\mathbf{p})+Z_-^n(x;\omega_-^n(\mathbf{p}),\mathbf{p})\bar{n}(x;\omega_-^n(\mathbf{p}),\mathbf{p}) \\ &+Z_-^a(x;\omega_-^a(\mathbf{p}),\mathbf{p})n(x;\omega_-^a(\mathbf{p}),\mathbf{p})+Z_+^a(x;\omega_+^a(\mathbf{p}),\mathbf{p})\bar{n}(x;\omega_+^a(\mathbf{p}),\mathbf{p})\Big]+\ldots.\end{aligned} \tag{5.6}$$

From Eqs. (5.5) and (5.6), we can read off that n is the number density of fermionic quasiparticle and $\bar{n}$ the number density of antifermionic quasiparticle. Undoing the narrow-width approximation yields further corrections to the physical number densities.

Let us turn to analyze the contributions from the nonleading part of $\hat{G}$ in Eq. (4.8). Inspection of Eqs. (5.3) and (5.4) with Eqs. (4.8) - (4.14) and (B.10) shows that all but $G_K^{(0)}$, Eq. (4.13) yield well-defined corrections to the physical number densities due to the medium effect. $G_K^{(0)}$ contains

$$g_R^{\pm} g_A^{\pm} = \frac{1}{\left[p^0 \mp p \mp \sigma_R^{\pm}\right] \left[p^0 \mp p \mp \left(\sigma_R^{\pm}\right)^*\right]}.$$

In the narrow-width approximation $\mathrm{Im}\sigma_{\pm}^{R} \to \mp 0^+$, $g_{\pm}^{R} g_{\pm}^{A}$ develops pinch singularity in a complex p^0-plane. Then the contributions of $G_K^{(0)}$ to Eqs. (5.3) and (5.4) diverge in this approximation. In practice, $\mathrm{Im}\sigma_R^{\pm}$ ($\propto g^2$) is a small quantity, so that the contribution, although not divergent, is large. This invalidates the perturbative scheme and a sort of "renormalization" is necessary for the number densities [3]. This observation leads us to introduce the condition $G_K^{(0)} = 0$ on the energy-shells:

$$\begin{aligned} &\left[\tilde{P}_{\pm} \cdot \partial_X f \mp i \left(\sigma_K^{\pm}\right)^{\mathrm{loop}} \pm \left\{\mathrm{Re}\sigma_R^{\pm},\ f\right\}_{\mathrm{P.B.}}\right]_{\text{on energy shells}} \\ &\quad = 0, \end{aligned} \tag{5.7}$$

where $\{...,\ ...\}_{\mathrm{P.B.}}$ is as in Eq. (1.4). This serves as determining equation for so far arbitrary f. (See below, for more details.) Now the above-mentioned large contributions, which turn out to the diverging contributions (due to the pinch singularities) in the narrow-width approximation, do not appear. Thus, the contributions from $G_K^{(0)}$ to Eqs. (5.3) and to (5.4) also yields well-defined corrections to the physical number densities.

B. Generalized Boltzmann equation

We are now in a position to disclose the physical meaning of Eq. (5.7). Taking up Eq. (5.7) with upper signs and setting p^0 on the energy-shell of the normal mode, $p^0 = \omega_+^n(X; \mathbf{p})$, we obtain, using the formulae in Appendix C,

$$\left(Z_+^n(X;P)\right)^{-1}\left[\frac{\partial}{\partial X^0}+\mathbf{v}_+^n(X;\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{X}}\right]n(X;P)$$
$$+\frac{\partial\mathrm{Re}\sigma_R^+(X;P)}{\partial X^\mu}\frac{\partial n(X;P)}{\partial P_\mu}=i\left(\sigma_K^+\right)^{\mathrm{loop}}(X;P).$$

Here $\mathbf{v}_+^n$ $(\equiv\partial\omega_+^n(X;\mathbf{p})/\partial\mathbf{p})$ is the group velocity of the mode [cf. Eq. (C.2)]. As will be shown in Appendix D, $i\left(\sigma_K^+\right)^{\mathrm{loop}}$ on the RHS is related to the net production rate, $\Gamma_{\mathrm{net\ p}}^n$, of the mode $p^0=\omega_+^n(X;\mathbf{p})$. Using Eqs. (D.1) and (C.3), we obtain

$$\left[\frac{\partial}{\partial X^0}+\mathbf{v}_+^n(X;\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{X}}\right]N(X;P)$$
$$+\frac{\partial\omega_+^n(X;\mathbf{p})}{\partial X^\mu}\frac{\partial N(X;P)}{\partial P_\mu}=\Gamma_{\mathrm{net}\ p}^n(X;\mathbf{p}).$$

This can further be rewritten in the form,

$$\left(\frac{d}{dX^0}+\mathbf{v}_+^n(X;\mathbf{p})\cdot\frac{d}{d\mathbf{X}}\right)n(X;\omega_+^n(X;\mathbf{p}),\hat{\mathbf{p}})$$
$$-\frac{\partial\omega_+^n(X;\mathbf{p})}{\partial\mathbf{X}}\frac{dn}{d\mathbf{P}}=\Gamma_{\mathrm{net}\ p}^n(X;\mathbf{p}). \tag{5.8}$$

Similarly, Eq. (5.7) (with upper signs) with $p_0=-\omega_+^a(X;-\mathbf{p})$ yields

$$\left(\frac{d}{dX^0}+\mathbf{v}_+^a(X;\mathbf{p})\cdot\frac{d}{d\mathbf{X}}\right)\bar{n}(X;\omega_+^a(X;\mathbf{p}),\mathbf{p})$$
$$-\frac{\partial\omega_+^a(X;\mathbf{p})}{\partial\mathbf{X}}\frac{d\bar{n}}{d\mathbf{P}}=\bar{\Gamma}_{\mathrm{net}\ p}^a(X;\mathbf{p}). \tag{5.9}$$

Equation (5.7) with lower signs yields, on the energy-shell $p_0=-\omega_-^n(X;-\mathbf{p})$,

$$\left(\frac{d}{dX^0}+\mathbf{v}_-^n(X;\mathbf{p})\cdot\frac{d}{d\mathbf{X}}\right)\bar{n}(X;\omega_-^n(X;\mathbf{p}),\mathbf{p})$$
$$-\frac{\partial\omega_-^n(X;\mathbf{p})}{\partial\mathbf{X}}\frac{d\bar{n}}{d\mathbf{P}}=\Gamma_{\mathrm{net}\ p}^n(X;\mathbf{p}), \tag{5.10}$$

and on the energy-shell $p_0=\omega_-^a(X;\mathbf{p})$,

$$\left(\frac{d}{dX^0}+\mathbf{v}_-^a(X;\mathbf{p})\cdot\frac{d}{d\mathbf{X}}\right)n(X;\omega_-^a(X;\mathbf{p}),\mathbf{p})$$
$$-\frac{\partial\omega_-^a(X;\mathbf{p})}{\partial\mathbf{X}}\frac{dn}{d\mathbf{P}}=\Gamma_{\mathrm{net}\ p}^a(X;\mathbf{p}). \tag{5.11}$$

Equations (5.8) - (5.11) are the generalized relativistic Boltzmann equation for (anti)fermionic quasiparticles.

VI. PERTURBATION THEORY

As has been discussed in the preceding section, the propagator in the physical-N scheme is free from the pinch singular term (in the narrow-width approximation) and then the perturbative calculation of some quantity yields "healthy" perturbative corrections. For constructing a concrete perturbative scheme, one more step is necessary.

To extend the condition (5.7) to off the energy-shell, we divide f into two pieces [cf. Eq. (2.9)]

$$\begin{aligned} f(X;P) &= \theta(-p^0) + \epsilon(p^0)N(X;P) \\ &= \tilde{f}(X;P) + f_0(X;P) \\ &= [\theta(-p^0) + \epsilon(p^0)\tilde{N}(X;P)] + \epsilon(p^0)N_0(X;P). \end{aligned} \tag{6.1}$$

f_0 (and then also $\tilde{f}$) is defined as follows: Let $\mathcal{R}_i(X;\mathbf{p})$ $(i = 1, 2, ...)$ be a region in a p^0-plane that includes ith energy-shell. We choose $\mathcal{R}_i(X;\mathbf{p})$, such that, for $i \neq j$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$. On each energy-shell, $N_0(X;P) = N(X;P)$, and, in whole p^0-region but $\mathcal{R}_i(X;\mathbf{p})$ $(i = 1, 2, ...)$, $N_0(X;P)$ vanishes. $\partial N_0(X;P)/\partial X$ and $\partial N_0(X;P)/\partial P$ exist and $N_0(X;P)$ obeys

$$\tilde{P}_\pm \cdot \partial_X N_0 \mp i \left(\sigma_K^\pm\right)^{\mathrm{loop}} \pm \left\{\mathrm{Re}\sigma_R^\pm,\ N_0\right\}_{\mathrm{P.B.}} = 0. \tag{6.2}$$

Then, $G_K^{(0)}$ is given by Eq. (4.13) with $\tilde{f}$ for f. It is obvious from the above construction that this $G_K^{(0)}$ does not possess pinch singularities in narrow-width approximation, and thus "healthy" perturbation theory is established.

It is worth mentioning that there is arbitrariness in the choice of the regions ,$\mathcal{R}_i(X;\mathbf{p})$ $(i = 1, 2, ...)$. Furthermore, the choice of the functional forms of $\tilde{f}$ and of f_0 is also arbitrary, provided that

$$\begin{aligned} &\tilde{f}(X^0 = X^0_{in}, \mathbf{X}; P) + f_0(X^0 = X^0_{in}, \mathbf{X}; P) \\ &\quad = f(X^0 = X^0_{in}, \mathbf{X}; P), \end{aligned}$$

where $f(X^0 = X^0_{in}, \mathbf{X}; P)$ is the initial data with X^0_{in} the initial time. As has been discussed at the end of Sec. III, these arbitrariness are not the matter.

To summarize, to the gradient approximation the (resummed) propagator $\hat{\mathcal{G}}$ of the theory is

$$\hat{\mathcal{G}} = \hat{G} + \hat{S}_1 + \hat{S}_2 + \hat{S}_3. \tag{6.3}$$

Here $\hat{S}_1$ - $\hat{S}_3$ are as in Eqs. (2.7), (A.1) - (A.2), and $\hat{G}$ is as in Eq. (4.8) provided that $G_K^{(0)}$ is given by Eq. (4.13) with $\tilde{f}$ for f. f consists of two pieces as in Eq. (6.1). f_0 $(= \epsilon(p^0)N_0)$ subjects to Eq. (6.2), which is to be solved under a given initial data. It is to be noted that $\hat{S}_1$ - $\hat{S}_3$ are the nonleading parts, so that, if one wants, for f in $\hat{S}_1$ - $\hat{S}_3$, one can substitute f_B, the solution to the "free Boltzmann equation," Eq. (2.13).

As in [5], determination of f or N proceeds order by order in perturbation theory, which we do not reproduce here.

The vertex factor and the initial correlations are the same as in standard CTP formalism, except that an additional two-point vertex $iL_c(x, y)\hat{A}_-$, Eq. (3.6), exists. It is to be noted that the two-point vertex $iL_c(x, y)\hat{A}_-$ has been built into $\hat{G}$ (cf. Eqs. (4.12) and (4.13) with Eqs. (B.9) and (B.7)) and is absent in the perturbative framework using $\hat{\mathcal{G}}$ in Eq. (6.3).

VII. SUMMARY AND DISCUSSION

In this paper we have dealt with out-of-equilibrium perturbation theory for massless Dirac fermions. The fermion propagator is constructed from first principles. Essentially only approximation we have employed is the so-called gradient approximation, so that the perturbative framework applies to the quasiuniform systems near equilibrium or the nonequilibrium quasistationary systems. The framework allows us to compute any reaction rates [11].

There comes out naturally the generalized Boltzmann equation (GBE) that describes the spacetime evolution of the number densities of quasiparticles, through which the evolution of the system is described.

Comparison of the present formalism with some earlier works has been made in [3]. We like to add here two related works [13,14], in which fermions are dealt with. In these papers, the GBE is derived in a traditional manner by starting with the SD equation (cf. Eq. (4.6)). As is mentioned in [3], the SD equation is nothing more than an equation that serves as resumming the self-energy part to makeup the resummed propagator, Eq. (4.8). Then, in order to derive the GBE, an additional input or condition is necessary. Our condition is Eq. (5.7). The additional input in [13,14] is, in our notation, an introduction of f' through

$$\begin{aligned} G_{12(21)}(X;P) &\simeq 2\pi i f'(X;P)\epsilon(p_0)\not{P}\sigma(P) \\ &= 2\pi i\left[\theta(p_0)N'(X;P)\right. \\ &\qquad \left. -\,\theta(-p_0)\left(1-\bar{N}'(X;P)\right)\right]\not{P}\sigma(P) \end{aligned} \tag{7.1}$$

with $\sigma(P) \simeq \delta(P^2)$. In [13], this is done on the basis of the quasiparticle picture, and, in [14], on the order-of-magnitude estimation of $\hat{\Sigma}$ in diagrammatic analysis. Substituting Eq. (7.1) into the Kadanoff-Baym equation, which is a part of the SD equation, the GBE for $N'(X;P)$ and $\bar{N}'(X;P)$ results. In contrast to the formalism in the present paper, no counter Lagrangian is explicitly introduced there and then the consistency check for the formalism seems to be necessary.

In [3,4], comparison has been made of the present formalism with nonequilibrium thermo field dynamics (NETFD) [12]. (See also [13]). We like to recapitulate here that, in NETFD, a counter action is introduced on the basis of the renormalizability argument. Imposition of the renormalization condition leads to the GBE.

The "derivation" in this paper of the GBE for a quark-gluon plasma (nonequilibrium QCD) is quite different from the traditional derivation (cf. [13,14]). What we have shown here is that the requirement of the absence of large contributions from the perturbative framework leads to the GBE. This means that the quasiparticles thus defined are the well-defined modes in the medium. Conversely, if we start with defining the quasiparticles such that their number density functions subject to the GBE, then, on the basis of them, well-

defined perturbation theory may be constructed. In [12], the GBE is derived by imposing the renormalization condition" for the propagator. Since the pinch singularity, which arises in the narrow-width limit in our formulation, is a singularity in momentum space, it is not immediately obvious how to translate this condition into (space)time representation, as adopted in [12]. Nevertheless, closer inspection of the structures of our formalism and of the NETFD tells us that our condition is in accord with on-shell renormalization condition in NETFD.

ACKNOWLEDGMENTS

This work was supported in part by a Grant-in-Aide for Scientific Research ((C)(2) (No. 12640287)) of the Ministry of Education, Science, Sports and Culture of Japan.

APPENDIX A: EXPRESSION FOR THE NONLEADING PART OF $\hat{S}(X;P)$

The nonleading parts, $\hat{S}_2$ and $\hat{S}_3$, of $\hat{S}$ read

$$\hat{S}_2(X;P) = -\left[\Delta_R(P) - \Delta_A(P)\right]\epsilon(p^0)\gamma_5 \not{P} N^{(-)}(P)\hat{A}_+, \tag{A.1}$$

$$\begin{aligned}\hat{S}_3(X;P) = -\left[\Delta_R(P) - \Delta_A(P)\right]\epsilon(p^0)\Bigg[&\left\{\sigma^{2j}p^j + \epsilon^{1ij}\sigma^{0i}p^j - \frac{\sigma^{3j}p^2p^j + \epsilon^{3ij}\sigma^{0i}p^1p^j}{p^0+p^3}\right\}\mathrm{Re}\mathcal{N}(P) \\ &+\left\{\sigma^{1j}p^j - \epsilon^{2ij}\sigma^{0i}p^j - \frac{\sigma^{3j}p^1p^j - \epsilon^{3ij}\sigma^{0i}p^2p^j}{p^0+p^3}\right\}\mathrm{Im}\mathcal{N}(P)\Bigg]\hat{A}_+,\end{aligned} \tag{A.2}$$

where $\epsilon^{123} = 1$, $\sigma^{\mu\nu} = i(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)/2$, and

$$\begin{aligned}N^{(-)}(X;P) &\equiv \frac{1}{2}\Big[\theta(p^0)\left\{N_{++}(X;|p^0|,\tilde{\mathbf{p}}) - N_{--}(X;|p^0|,\tilde{\mathbf{p}})\right\} \\ &\qquad -\theta(-p^0)\left\{\bar{N}_{++}(X;|p^0|,-\tilde{\mathbf{p}}) - \bar{N}_{--}(X;|p^0|,-\tilde{\mathbf{p}})\right\}\Big] \\ &\simeq N^{(-)}(P), \\ \mathcal{N}(P) &\equiv \theta(p^0)N_{-+}(|p_0|,\tilde{\mathbf{p}}) + \theta(-p^0)\bar{N}_{-+}(|p_0|,-\tilde{\mathbf{p}}).\end{aligned}$$

As mentioned at the beginning of Sec. IIB, we have assumed that $\hat{S}_2$ and $\hat{S}_3$ are, at most, of the same order of magnitude as the gradient term $\hat{S}_1$, and then X-dependence of $N^{(-)}$

and $\mathcal{N}$ have been ignored. It can readily be seen that, to the gradient approximation, $i\hat{\tau}_3\partial\!\!\!/_x\hat{S}_2(x,y) = i\hat{\tau}_3\partial\!\!\!/_x\hat{S}_3(x,y) = 0$ holds. One can see from Eq. (A.2) that $\hat{S}_3$ does not contribute to the unpolarized quantities.

For a charge-conjugation-invariant system, $N^{(-)}(P) = N^{(-)}(-P)$ and $\mathcal{N}(P) = \mathcal{N}(-P)$ hold.

APPENDIX B: SOLVING THE SD EQUATION, EQ. (4.6)

Multiplying $\hat{B}_L^{-1}\cdot = \hat{\tau}_3\hat{B}_R\hat{\tau}_3\cdot$ ($\cdot\hat{B}_R^{-1} = \cdot\hat{\tau}_3\hat{B}_L\hat{\tau}_3$) [cf. Eq. (4.3)] from the left (right) of each term in Eq. (4.6), we obtain

$$\underline{\hat{G}} = \underline{\hat{\mathcal{S}}} + \underline{\hat{\mathcal{S}}}\cdot\underline{\hat{\Sigma}}\cdot\underline{\hat{G}} = \underline{\hat{\mathcal{S}}} + \underline{\hat{G}}\cdot\underline{\hat{\Sigma}}\cdot\underline{\hat{\mathcal{S}}}, \tag{B.1}$$

$$\underline{\hat{G}} \equiv \hat{B}_L^{-1}\cdot\hat{G}\cdot\hat{B}_R^{-1}, \tag{B.2}$$

$$\underline{\hat{\mathcal{S}}} \equiv \hat{B}_L^{-1}\cdot\hat{\mathcal{S}}\cdot\hat{B}_R^{-1} = \begin{pmatrix} i\partial\!\!\!/\Delta_R & S_K \\ 0 & -i\partial\!\!\!/\Delta_A \end{pmatrix}, \tag{B.3}$$

$$\underline{\hat{\Sigma}} \equiv \hat{B}_R\cdot\hat{\Sigma}\cdot\hat{B}_L = \begin{pmatrix} \Sigma_R & \Sigma_K \\ 0 & -\Sigma_A \end{pmatrix}. \tag{B.4}$$

Here $\Sigma_{R(A)} = \Sigma_{11} + \Sigma_{12(21)}$, and S_K and Σ_K are the inverse Wigner transforms of

$$\begin{aligned} S_K(X;P) = i\Big[&\partial\!\!\!/_X f(X;P) \\ &- \frac{\gamma^0+\gamma^3}{p^0+p^3}P\cdot\partial_X f(X;P)\Big]\frac{\mathbf{P}}{P^2}, \end{aligned} \tag{B.5}$$

and of

$$\begin{aligned} \Sigma_K(X;P) = [1-f(X;P)]&\Sigma_{12}(X;P) \\ &+f(X;P)\Sigma_{21}(X;P) \\ &+\frac{i}{2}\{\Sigma_R+\Sigma_A,\ f\}_{\text{P.B.}}, \end{aligned} \tag{B.6}$$

respectively. Here $\{..., ...\}_{\rm P.B.}$ is as in Eq. (1.4). In obtaining Eq. (B.5) [Eq. (B.6)], use has been made of Eq. (4.5) [Eq. (4.7)]. Although the last term in Eq. (B.6) may be dropped to the approximation under consideration, we have kept it.

From the above definitions of $\Sigma_{R(A)}$, Σ_K, and of $\hat{\Sigma}^{(c)}$ (Eq. (3.6)), we have

$$\begin{aligned} \Sigma_R^{(c)} &= \Sigma_A^{(c)} = 0, \\ \Sigma_K^{(c)}(X;P) &= i\frac{\gamma^0+\gamma^3}{p^0+p^3} P\cdot\partial_X f(X;P). \end{aligned} \tag{B.7}$$

As seen from Eqs. (B.3) and (B.4), $\hat{\mathcal{S}}$ and $\hat{\Sigma}$ are triangular matrices, so that Eq. (B.1) may easily be solved to yield

$$\begin{aligned} \underline{\hat{G}} &= \begin{pmatrix} G_R & G'_K \\ 0 & -G_A \end{pmatrix}, \\ G_{R(A)} &= \left[i\partial\!\!\!/ \Delta_{R(A)} - \Sigma_{R(A)} \right]^{-1}, \end{aligned} \tag{B.8}$$

$$G'_K = G_R \cdot \left[\left(i\partial\!\!\!/ \Delta_R \right)^{-1} \cdot S_K \cdot \left(i\partial\!\!\!/ \Delta_A \right)^{-1} - \Sigma_K \right] \cdot G_A. \tag{B.9}$$

Substituting this back into $\hat{G} = \hat{B}_L \cdot \underline{\hat{G}} \cdot \hat{B}_R$, Eq. (B.2), we obtain, after Wigner transformation, Eq. (4.8) with

$$G_K(X;P) = G'_K(X;P) + \frac{i}{2}\left\{ G_R + G_A,\ f \right\}_{\rm P.B.}.$$

From Eq. (B.8), we obtain, after some manipulation, Eq. (4.9) with Eq. (4.10) and with

$$\begin{aligned} G^{(1)}_{R(A)}(X;P) = \frac{1}{4p} & \left[g^+_{R(A)}(X;P) - g^-_{R(A)}(X;P) \right] \\ & \times \sum_{\tau=\pm} \left[\gamma_5 \vec{\gamma} \cdot \left(\tilde{\mathbf{p}} \times \nabla_X \sigma^\tau_{R(A)}(X;P) \right) \right. \\ & \qquad \left. \times g^\tau_{R(A)}(X;P) \right], \end{aligned} \tag{B.10}$$

where $g^{\pm}_{R(A)}$ is as in Eq. (4.11) and $\sigma^\tau_{R(A)} = \sigma^\tau_{11} + \sigma^\tau_{12(21)}$.

APPENDIX C: ON THE ENERGY SHELL

Here we display some formulae, which hold on the energy-shells of quasiparticles [cf. Eqs. (5.1) and (5.2) with Eqs. (4.10) and (4.11)]. In most formulae in this Appendix, we drop the argument X.

Normal modes:

We define the wave-function renormalization factors through taking derivative of Eq. (5.1) with respect to p^0:

$$\left(Z_\pm^n(\omega_\pm^n(\mathbf{p}),\mathbf{p})\right)^{-1} = 1 \mp \frac{\partial\,\mathrm{Re}\,\sigma_R^\pm(p^0,\pm\mathbf{p})}{\partial p^0}\Big|_{p^0\,=\,\pm\omega_\pm^n(\mathbf{p})}. \tag{C.1}$$

The group velocities of the modes are obtained from the definition (5.1),

$$\begin{aligned}\mathbf{v}_\pm^n(\mathbf{p}) &\equiv \frac{d\omega_\pm^n(\mathbf{p})}{d\mathbf{p}} \\ &= Z_\pm^n(\omega_\pm^n(\mathbf{p}),\mathbf{p}) \\ &\quad\times\left[\hat{\mathbf{p}} + \frac{\partial\,\mathrm{Re}\,\sigma_R^\pm(p^0,\pm\mathbf{p})}{\partial\mathbf{p}}\Big|_{p^0\,=\,\pm\omega_\pm^n(\mathbf{p})}\right].\end{aligned} \tag{C.2}$$

By differentiating Eq. (5.1) with respect to X, we obtain

$$\frac{\partial\omega_+^n(X;\mathbf{p})}{\partial X} = Z_+^n(\omega_+^n(\mathbf{p}),\mathbf{p})\frac{\partial\,\mathrm{Re}\,\sigma_R^+(X;\omega_+^n(X;\mathbf{p}),\mathbf{p})}{\partial X}. \tag{C.3}$$

Abnormal modes:

$$\begin{aligned}\left(Z_\pm^a(\omega_\pm^a(\mathbf{p}),\mathbf{p})\right)^{-1} &= 1 \mp \frac{\partial\,\mathrm{Re}\,\sigma_R^\pm(p^0,\mp\mathbf{p})}{\partial p^0}\Big|_{p^0\,=\,\mp\omega_\pm^a(\mathbf{p})}, \\ \mathbf{v}_\pm^a(\mathbf{p}) &\equiv \frac{d\omega_\pm^a(\mathbf{p})}{d\mathbf{p}} \\ &= -Z_\pm^a(\omega_\pm^a(\mathbf{p}),\mathbf{p}) \\ &\quad\times\left[\hat{\mathbf{p}} + \frac{\partial\,\mathrm{Re}\,\sigma_R^\pm(p^0,\mp\mathbf{p})}{\partial\mathbf{p}}\Big|_{p^0\,=\,\mp\omega_\pm^a(\mathbf{p})}\right].\end{aligned} \tag{C.4}$$

APPENDIX D: NET PRODUCTION RATES

$p^0 = \omega_\pm^{n/a}(\mathbf{p})$ *fermionic modes*:

From Eqs. (4.10), (5.1), and (5.2), we see that the projection operators onto $p^0 = \omega_+^n$ ($p^0 = \omega_-^a$) mode is $\tilde{\not{P}}_+$ ($\tilde{\not{P}}_-$). Then, the production and decay rates are written, in respective order, as [7,3]

$$\begin{aligned}
\Gamma_p^{n/a}(\mathbf{p}) &= \frac{i}{4} Z_\pm^{n/a}(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p}) \\
&\quad \times \mathrm{Tr}\left[\left(\Sigma_{12}(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}} \tilde{\not{P}}_\pm\right] \\
&= \pm i Z_\pm^{n/a} \left(\sigma_{12}^\pm(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}}, \\
\Gamma_d^{n/a}(X; \mathbf{p}) &= -\frac{i}{4} Z_\pm^{n/a}(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p}) \\
&\quad \times \mathrm{Tr}\left[\left(\Sigma_{21}(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}} \tilde{\not{P}}_\pm\right] \\
&= \mp i Z_\pm^{n/a} \left(\sigma_{21}^\pm(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}},
\end{aligned}$$

where Z's are the wave-function renormalization factor, Eqs. (C.1) and (C.4). Thus, the net production rate is

$$\begin{aligned}
\Gamma_{\mathrm{net}\, p}^{n/a}(\mathbf{p}) &= [1 - n(\omega_\pm^{n/a}(\mathbf{p}), \tilde{\mathbf{p}})] \Gamma_p^{n/a}(X; \mathbf{p}) \\
&\quad - n(\omega_\pm^{n/a}(\mathbf{p}), \tilde{\mathbf{p}}) \Gamma_d^{n/a}(\mathbf{p}) \\
&= \pm i Z_\pm^{n/a}(\omega_\pm^{n/a}(\mathbf{p}), \mathbf{p}) \\
&\quad \times \left(\sigma_K^\pm(\omega_\pm^{n/a}(X; \mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}},
\end{aligned} \tag{D.1}$$

where $\left(\sigma_K^\pm\right)^{\mathrm{loop}}$ is as in Eq. (4.15).

$p^0 = -\omega_\mp^{n/a}(-\mathbf{p})$ *antifermionic modes*:

$$\begin{aligned}
\overline{\Gamma}_p^{n/a}(-\mathbf{p}) &= -\frac{i}{4} Z_\mp^{n/a}(\omega_\mp^{n/a}(-\mathbf{p}), -\mathbf{p}) \\
&\quad \times \mathrm{Tr}\left[\left(\Sigma_{21}(-\omega_\mp^{n/a}(-\mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}} \tilde{\not{P}}_\mp\right] \\
&= \pm i Z_\mp^{n/a} \left(\sigma_{21}^\mp(-\omega_\mp^{n/a}(-\mathbf{p}), \mathbf{p})\right)^{\mathrm{loop}}, \\
\overline{\Gamma}_d^{n/a}(-\mathbf{p}) &= \frac{i}{4} Z_\mp^{n/a}(\omega_\mp^{n/a}(-\mathbf{p}), -\mathbf{p})
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{Tr}\left[\left(\Sigma_{12}(-\omega_{\mp}^{n/a}(-\mathbf{p}),\mathbf{p})\right)^{\mathrm{loop}}\tilde{\not{P}}_{\mp}\right] \\
&= \mp i Z_{\mp}^{n/a}\left(\sigma_{12}^{\mp}(-\omega_{\mp}^{n/a}(-\mathbf{p}),\mathbf{p})\right)^{\mathrm{loop}}, \\
\overline{\Gamma}_{\mathrm{net}\ p}^{n/a}(-\mathbf{p}) &= [1-\bar{n}(\omega_{\mp}^{n/a}(-\mathbf{p}),-\mathbf{p})]\overline{\Gamma}_{p}^{n/a}(-\mathbf{p}) \\
&\quad -\bar{n}(\omega_{\mp}^{n/a}(-\mathbf{p}),-\mathbf{p})\overline{\Gamma}_{d}^{n/a}(-\mathbf{p}) \\
&= \pm i Z_{\mp}^{n/a}(\omega_{\mp}^{n/a}(-\mathbf{p}),-\mathbf{p}) \\
&\qquad \times\left(\sigma_{K}^{\mp}(-\omega_{\mp}^{n/a}(-\mathbf{p}),\mathbf{p})\right)^{\mathrm{loop}}.
\end{aligned}
$$

REFERENCES

[1] J. D. Bjorken, *Acta Phys. Polon. B* **28**, 2773 (1997); See also, e.g., K. Rajagopal, in Quark-Gluon Plasma 2, edited by R. C. Hwa (World Scientific, Singapore, 1995), p.484.

[2] M. Le Bellac, "Thermal Field Theory," Cambridge University Press, Cambridge, England, 1996.

[3] A. Niégawa, Prog. Theor. Phys. **102**, 1 (1999).

[4] A. Niégawa, Phys. Rev. D **62**, 125004 (2000).

[5] A. Niégawa, Phys. Rev. D **64**, 036004 (2001).

[6] J. Schwinger, J. Math. Phys. **2**, 407 (1961); L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) (Sov. Phys. JETP **20**, 1018 (1965)); R. A. Craig, J. Math. Phys. **9**, 605 (1968); J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).

[7] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[8] N. P. Landsman and Ch. G. van Weert, Phys. Rep. **145**, 141 (1987).

[9] E. Calzetta and B. L. Hu, Phys. Rev. D **37**, 2878 (1988).

[10] M. Le Bellac and H. Mabilat, Z. Phys. C **75**, 137 (1997).

[11] A. Niégawa, K. Okano, and H. Ozaki, Phys. Rev. D **61**, 056004 (2000). For the case of equilibrium systems, see, A. Niégawa, *ibid.* **57**, 1379 (1998).

[12] H. Umezawa, "Advanced Field Theory — Micro, Macro, and Thermal Physics." AIP, New York, 1993; H. Chu and H. Umezawa, Int. J. Mod. Phys. **A9**, 1703 and 2363 (1994) and earlier papers quoted therein.

[13] P. A. Henning, Phys. Rep. **253**, 235 (1995).

[14] J.-P. Blaizot and E. Iancu, hep-ph/0101103. See also, J.-P. Blaizot and E. Iancu, Nucl. Phys. **B557**, 183 (1999).